INTEGRAL CALCULUS

Project

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Class: S1 Mathematics

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Introduction

Calculus is the branch of Mathematics, which was developed by Newton and Leibniz deals with the rate of change. In mathematics, an integral is the continuous analog of a sum, which is used to calculate areas, volumes, and their generalizations. Integration, the process of computing an integral, is one of the two fundamental operations of calculus, the other being differentiation. Differential calculus deals with the rate of change of one quantity with respect to another. The process of differentiation is used for finding derivatives and differentials of a function. Whereas Integral calculus is the reverse process of differentiation, which is to find the functions from their derivatives. Thus the process of finding anti derivatives is called integration. So, integration and differentiation are closely related.

The process of integration is used

i. To find the function whose derivative is given
ii. To find the area bounded by the function under certain circumstances

These two processes lead to 2 forms of integrals, i.e., definite and indefinite integrals which together constitute integral calculus
In integration whether the object be summation or anti-differentiation, the sign $\int$, an elongated S, the first letter of

the word 'sum' is most generally used to indicate the process of integration.
Therefore, ∫f(x) dx is read the integral of f(x) with respect to x.

eg: If d/dx g(x) = f(x)
Then ∫f(x) dx = g(x) + c
where c is called the constant of integration

Some Standard Results of Integration

A list of some standard results by using the derivative of some well-known functions is given below:

i. $\int dx = x + c$

ii. $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$

iii. $\int \frac{1}{x} dx = \log x + c$

iv. $e^x \, dx = e^x + c$

v. $\int a^x dx = \frac{a^x}{\log a} + c$

vi. $\int \sin x \, dx = -\cos x + c$

vii. $\int \cos x \, dx = \sin x + c$

viii. $\int \sec^2 x \, dx = \tan x + c$

ix. $\int csc^2 x\, dx = -\cot x + c$

x. $\int \sec x \tan x\, dx = \sec x + c$

xi. $\int \csc x \cot x dx = -\csc x$+c

xii. $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$

xiii. $\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$

xiv. $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + c$

Examples

- $\int 5x\, dx$

$$\int 5\text{x}\, dx = 5 \times \frac{x^2}{2} + c$$

$= \frac{5}{2}x^2 + c$

- $\int (2x + 3)dx$

$\int (2\text{x} + 3)\text{dx} = 2 \times \frac{x^2}{2} + 3\text{x+c}$

$= x^2 + 3x + c$

- $\int sinx\, dx$

$$= \int sinx\, dx = -\cos x + c$$

- $\int 4x^2 - 7x + 6$

$$\int 4x^2 - 7x + 6 = \int 4x^2 - \int 7x + \int 6$$

$$= 4\times\frac{x^3}{3} - 7\times\frac{x^2}{2} + 6x + c$$

$$= \frac{4}{3}x^3 - \frac{7}{2}x^2 + 6x + c$$

Methods of Integration

The following are the four principal methods of integration:

(i) Integration by substitution;
(ii) Integration by parts;
(iii) Integration by successive reductions;
(iv) Integration by partial fraction.

i. Integration by Substitution

Substituting a new suitable variable for the given independent variable and integrating with respect to the substituted variable can often facilitate Integration.

Example:

1. $\int (ax+b)^5 dx$

Put ax+b = u

$\frac{du}{dx}$ =a

dx= $\frac{1}{a} du$

$\int (ax+b)^5 dx = \int u^5 \frac{1}{a} du$

$= \frac{1}{a} \int u^5 du$

$= \frac{u^6}{6a} + c$

$= \frac{(ax+b)^5}{6a} + c$

ii. Integration by Parts

Integration by parts is a special method that can be applied in finding the integrals of a product of two integrable functions.

Integral of the product of two functions
= 1st function × integral of the 2nd – integral of (differential of 1st × integral of 2nd).

Example:

1. $\int x^2 \log x \, dx$

$$= \log x \times \int x^2 - \int \left[\frac{d(\log x)}{dx} \times \int x^2\right]$$

$$= \log x \frac{x^3}{3} - \int \left[\frac{1}{x} \times \frac{x^3}{3}\right]$$

$$= \frac{x^3}{3} \log x - \int \frac{x^2}{3}$$

$$= \frac{x^3}{3} \log x - \frac{x^3}{9} + c$$

iii. **Integration by successive reductions**

Any formula expressing a given integral in terms of another that is simpler than it, is called a reduction formula for the given integrals. In practice, however, the reduction formula for a given integral means that the integral belongs to class of integrals such that it can be expressed in terms of one or more integrals or lower orders belonging to the same class; by successive application of the formula, we arrive at integrals which can be easily integrated and hence the given integral can be evaluated.

<u>Example:</u>

$$\int x^2 e^{2x} dx$$

$$\int x^2e^{2x}\,dx = x^2\int e^{2x} - \int\left[\frac{d(x^2)}{dx}\times\int e^{2x}dx\right]dx$$

$$= x^2\times\frac{e^{2x}}{2} - \int\left(2x\times\frac{e^{2x}}{2}\right)dx$$

$$= \frac{x^2e^{2x}}{2} - \int xe^{2x}dx$$

$$= \frac{x^2e^{2x}}{2} - \left\{x\times\int e^{2x} - \int\frac{d(x)}{dx}\times \int(e^{2x}dx)dx\right\}$$

$$= \frac{x^2e^{2x}}{2} - \left\{\frac{xe^{2x}}{2} - \int 1\times\frac{e^{2x}}{2}dx\right\}$$

$$= \frac{x^2e^{2x}}{2} - \frac{1}{2}\times\frac{e^{2x}}{2} + c$$

$$= \frac{x^2e^{2x}}{2} - \frac{e^{2x}}{4} + c$$

$$= \frac{e^{2x}}{2}\left[x^2 - \frac{1}{2}\right] + c$$

iv. **Integration by partial fraction.**

Rational functions have the form of a quotient of two polynomials. Many rational functions exist which cannot be integrated by the rules of integration presented earlier. When these occur, one possibility is that the rational function can be restated in an equivalent form consisting of more elementary functions and then each of the component fractions can be easily integrated separately.

Example:

$$\int \frac{x+3}{x^2+3x+2}$$

$$\int \frac{x+3}{x^2+3x+2} = \int \frac{x+3}{(x+1)(x+2)}$$

Let $\frac{x+3}{(x+1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$

× both sides by (x+1)(x+2)

$(x+3) = A(x+2) + B(x+1)$

$A = 2, \quad B = -1$

$\therefore \int \frac{2}{(x+1)} - \frac{1}{(x+2)} dx$

$= 2\log(x+1) - \log(x+2) + c$

Definite Integrals

In Geometry and other application areas of integral calculus, it becomes necessary to find the difference in the values of an integral f(x) for two assigned values of the independent variable x, say, a, b, (a < b), where a and b are two real numbers. The difference is called the definite integral of f(x) over the domain (a, b) and is denoted by;

$$\int_a^b f(x)$$

If g(x) is the integral of f(x)
Then $\int_a^b f(x)dx = [g(x)]_a^b = g(b) - g(a)$

Here $\int_a^b$ f(x)dx is called the definite integral, as the constant of integration does not appear in it.

Properties of Definite Integrals

Some properties of definite integrals are given below;

i. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

ii. $\int_a^a f(x)dx = 0$

iii. $\int_a^b kf(x)\, dx = k\int_a^b f(x)dx$

iv. $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

v. $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$

Examples:

- $\int_a^b cx^2 dx$

$=c\int_a^b x^2 dx = c\left[\frac{x^3}{3}\right]_a^b$

$=\frac{c}{3}(a^3 - b^3)$

- $\int_1^4 \frac{1}{\sqrt{x}}dx$

$= \int_1^4 x^{-\frac{1}{2}}dx$

$= 2\left[x^{\frac{1}{2}}\right]_1^4$

$= 2\left[4^{\frac{1}{2}} - 1^{\frac{1}{2}}\right]$

$= 2(2\text{-}1)$

$=2$

- $\int_{-4}^4 (2x^2 - 4x + 2)dx$

$$= 2\int_0^4 (2x^2 - 4x + 2)$$
$$= 2\left[2\frac{x^3}{3} - \frac{4x^2}{2} + 2x\right]_0^4$$
$$= 4\left[\frac{x^3}{3} - \frac{2x^2}{2} + x\right]_0^4$$
$$= 4\left[\frac{4^3}{3} - 2 \times \frac{4^2}{2} + 4\right]$$
$$= 4\left[\frac{64}{3} - 16 + 4\right]$$
$$= 4 \times \frac{28}{3}$$
$$= \frac{112}{3}$$

⅄ $\int_0^\pi \sin x \, dx$

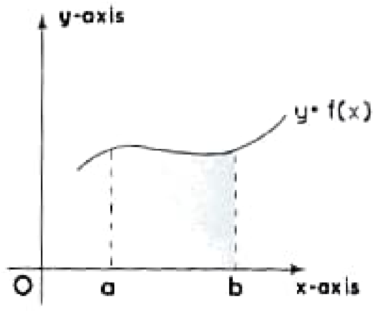
$$= [-\cos x]_0^\pi$$
$$= -[\cos\pi - \cos 0]$$
$$= \cos 0 - \cos\pi$$
$$= 1-(-1)$$
$$= 1+1=2$$

Applications of integration

1. To find the area under a curve

The area under the curve can be calculated through three simple steps. First, we need to know the equation of the curve (y = f(x)), the limits across which the area is to be calculated, and the axis enclosing the area. Secondly, we have to find the integration of the curve. Finally, we need to apply the upper limit and lower limit to the integral answer and take the difference to obtain the area under the curve.

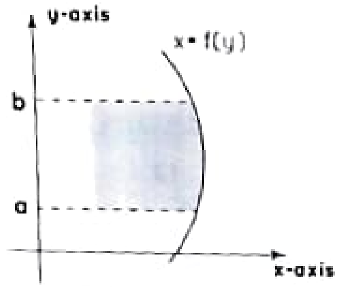
$$\text{Area} = \int_a^b y\,dx$$
$$= \int_a^b f(x)dx; \qquad \text{If } \int f(x)dx = g(x)$$
$$= g(x)_a^b$$
$$= g(b) - g(a)$$



This is to find the area under a curve with respect to X axis

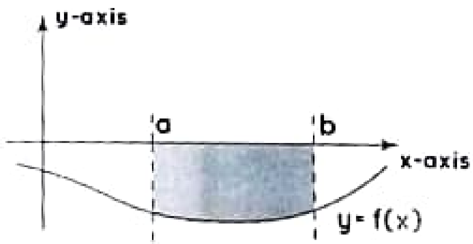
For finding the area under a curve with respect to Y axis, the eqn will be $x = f(y)$

$$\therefore \text{area} = \int_a^b x\,dy$$
$$= \int_a^b f(y)\,dy$$



Area below the axis: The area of the curve below the axis is a negative value and hence the modulus of the area is taken. The area of the curve y = f(x) below the x-axis and bounded by the x-axis is obtained by taking the limits a and b. The formula for the area above the curve and the x-axis is as follows

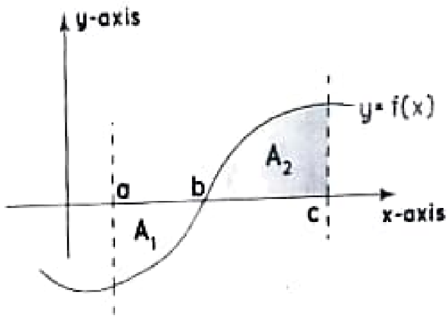
$$A = \left|\int_a^b f(x)\,dx\right|$$



Area above and below the axis: The area of the curve which is partly below the axis and partly above the axis is divided into two areas and separately calculated. The area under the axis is negative, and hence a modulus of the area is taken. Therefore the overall area is equal to the sum of the two areas

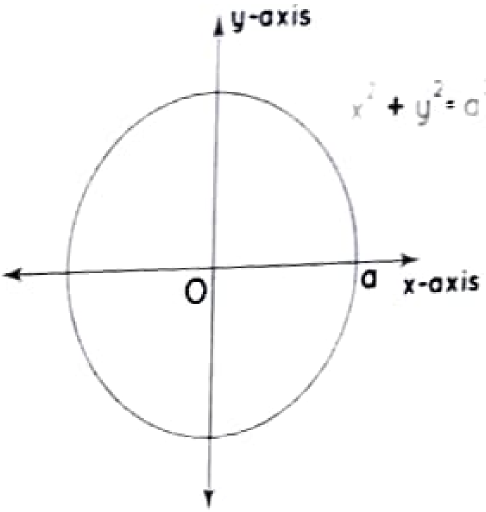
$$A = |A_1| + A_2$$

$$A = \left|\int_a^b f(x)dx\right| + \int_b^c f(x)dx$$



2. Area under a circle

The area of the circle is calculated by first calculating the area of the part of the circle in the first quadrant. Here the equation of the circle $x^2 + y^2 = a^2$ is changed to an equation of a curve as $y = \sqrt{(a^2 - x^2)}$. This equation of the curve is used to find the area with respect to the x-axis and the limits from 0 to a.



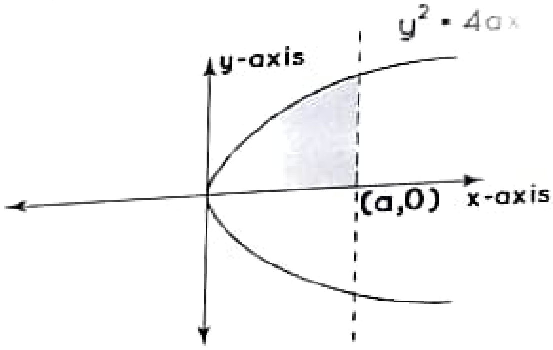
The area of the circle is four times the area of the quadrant of the circle. The area of the quadrant is calculated by integrating the equation of the curve across the limits in the first quadrant

$$\begin{aligned}
\text{Area} &= 4\int_0^a y\,dx \\
&= 4\int_0^a \sqrt{a^2-y^2}\,dx \\
&= 4\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a \\
&= 4\left[(\frac{a}{2}\times 0+\frac{a^2}{2}\sin^{-1}1)-0\right] \\
&= 4\left[\frac{a^2}{2}\times\frac{\pi}{2}\right] \\
&= 4\left(\frac{\pi a^2}{4}\right) \\
&= \pi a^2
\end{aligned}$$

Hence the area of the circle is πa^2 *square units.*

3. Area under a parabola

A parabola has an axis that divides the parabola into two symmetric parts. Here we take a parabola that is symmetric along the x-axis and has an equation $y^2 = 4ax$. This can be transformed as $y = \sqrt{(4ax)}$. We first find the area of the parabola in the first quadrant with respect to the x-axis and along the limits from 0 to a. Here we integrate the equation within the boundary and double it, to obtain the area of the whole parabola. The derivations for the area of the parabola is as follows.

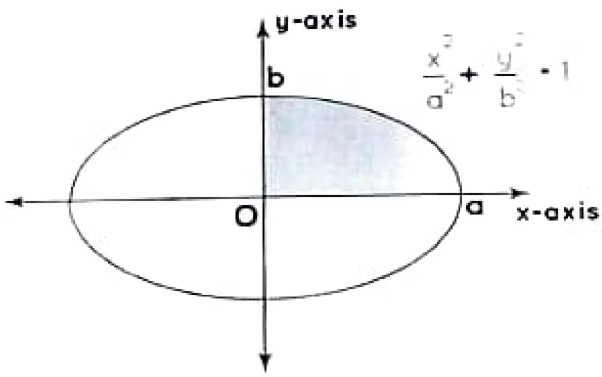


$$A = 2\int_0^a y\,dx$$
$$= 2\int_0^a \sqrt{4ax}\,dx$$
$$= 4\int_0^a \sqrt{ax}\,dx$$
$$= 4\sqrt{a}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^a$$
$$= \frac{8}{3}\sqrt{a}\left[a^{\frac{3}{2}} - 0\right]$$
$$= \frac{8a^2}{3}$$

Therefore the area under the curve enclosed by the parabola is $\frac{8a^2}{3}$ square units.

4. Area under an ellipse

The equation of the ellipse with the major axis of 2a and a minor axis of 2b is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. This equation can be transformed in the form as $y = \frac{b}{a} . \sqrt{(a^2 - x^2)}$. Here we calculate the area bounded by the ellipse in the first coordinate and with the x-axis, and further multiply it with 4 to obtain the area of the ellipse. The boundary limits taken on the x-axis is from 0 to a. The calculations for the area of the ellipse are as follows.



$A = 4 \int_0^a y \, dx$