FUNCTIONAL ANALYSIS

PROJECT

Submitted to University of Kerala in partial fulfillment of the requirements for the award of the Degree of Bachelor of Science
in
Mathematics

By

GOPIKA KRISHNA S	22020100005
PIYUSHA ULLAS	22020100007
SAM ROJIYA S	22020100010
ANJALI SATHEESH	22020100019
RASHIDHA R	22020100029
TALETHA LISA JOHN	22020100035

Under the Guidance of
Mrs. Renjini Raveendran P
Assistant Professor in Mathematics
All Saints' College

ALL SAINTS' COLLEGE
THIRUVANANTHAPURAM
2023

CERTIFICATE

This is to certify that the project "**FUNCTIONAL ANALYSIS**" is based on the work carried out by *GOPIKA KRISHNA S, PIYUSHA ULLAS, SAM ROJIYA S, ANJALI SATHEESH, RASHIDHA R and TALETHA LISA JOHN* under the guidance of *Mrs. RENJINI RAVEENDRAN P*, Assistant Professor in Mathematics, All Saints' College, Thiruvananthapuram and no part of this work has formed the basis for the award of any degree or diploma to any other University.

March, 2023
Thiruvanathapuram

For Renjini
Shiby Joseph

Teacher in charge

Head of the Department

HEAD OF THE DEPT. OF MATHEMATICS
ALL SAINTS' COLLEGE
TRIVANDRUM - 7

18/05/2023
Dr. John K. Rajan

18/5/23
Rajeswari B

ACKNOWLEDGEMENT

We are deeply indebted to our supervisor *Mrs. Renjini Raveendran P*, Assistant Professor in Mathematics, All Saints' College, Thiruvananthapuram, for her inspiring guidance and constant help throughout the preparation of this project work. We also extend our thanks to all our friends who helped us in one or other way to carry out this work.
Last, but not the least, we thank God Almighty for His helping hand.

GOPIKA KRISHNA S
PIYUSHA ULLAS
SAM ROJIYA S
ANJALI SATHEESH
RASHIDHA R
TALETHA LISA JOHN

CONTENTS

INTRODUCTION

Functional analysis is the branch of mathematics where vector spaces and operators on them are in focus. In linear algebra, the discussion is about finite dimensional vector spaces over any field of scalars. The functions are linear mappings which can be viewed as matrices with scalar entries. If the functions are mappings from a vector space to itself, the functions are called operators and they are represented by square matrices. In functional analysis, the vector spaces are in general infinite dimensional and not all operators on them can be represented by matrices. Hence the theory becomes more complicated, but nonetheless there are many similarities. Functional analysis has its origin in ordinary and partial differential equations, and in the beginning of the 20th century it started to form a discipline of its own via integral equations. However, for a long time there were doubts wether the mathematical theory was rich enough. Despite the efforts of many prominent mathematicians, it was not sure if there were sufficiently many functionals to support a good theory, and it was not until 1920 that the question was finally settled with the celebrated Hahn–Banach theorem. Seen from the modern point of view, functional analysis can be considered as a generalization of linear algebra. However, from a historical point of view, the theory of linear algebra was not developed enough to provide a basis for functional analysis at its time of creation. Thus, to study the history of functional analysis we need to investigate which concepts of mathematics that needed to be completed in order to get a theory rigorous enough to support it. Those concepts turn out to be functions, limits and set theory. For a long time, the definition of a function was due to Euler in his Introductio in Analysin Infinitorum from 1748 which read: *"A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities."*. For the purpose of this report, it is enough to say that the entire focus of this definition is on the function itself, and the properties of this particular function. What lead to the success of functional analysis was that the focus was lifted from the function, and shifted to the algebraic properties of sets of functions – The algebraization of analysis. The process of algebraization led mathematicians to study sets of functions where the functions are nothing more than abstract points in the set.

This Project contains five chapters. First chapter deals with the preliminaries which includes some basic definitions and examples of metric spaces, topology,open and closed ball,vector spaces, linear functionals, linear metric space etc. Second Chapter deals with Normed Linear Spaces and Bounded linear transformations. It includes the definition of norm, normed linear space,seminorm,p-norm, Cauchy sequence, Banach space, Bounded Linear Transformation, some propositions and one corollary.The theory of normed vector spaces was developed in the 1920-s by Banach, Hahn and Wiener (concrete functional spaces with a norm, without the name, were studied before that). Their use became a standard tool Banach's 1932 book Théorie des opérations linéaires (Theory of Linear Operations, 1932), which flashed out the role of completeness and systematically developed the theory of linear operators on complete (Banach) spaces.

Third Chapter deals with the famous Hahn-Banach Theorem. This is one of the most fundamental theorems in functional analysis and is due to Hahn and Banach. It yields the existence of non-trivial continuous linear functionals on a normed linear space, a basic result necessary for the development of a large portion of functional analysis.This chapter also includes some more theorems.The Hahn–Banach theorem is a central tool in functional analysis. It arose from attempts to solve infinite systems of linear equations. This is needed to solve problems such as the moment problem, whereby given all the potential moments of a function one must determine if a function having these moments exists, and, if so, find it in terms of those moments. Another such problem is the Fourier cosine series problem, whereby given all the potential Fourier cosine coefficients one must determine if a function having those coefficients exists, and, again, find it if so.

Fourth chapter deals with Closed Graph Theorem and Open Mapping theorem. The closed graph theorem is an important result in functional analysis that guarantees that a closed linear operator is continuous under certain conditions.The open mapping theorem asserts that certain continuous linear transformations between Banach spaces map open sets into open sets. It also includes some more theorems and examples.

The last chapter deals with Banach-Steinhaus Theorem which is another famous theorem of Functional Analysis. It is one of the most celebrated results in the theory of Banach spaces. This theorem has various important applications. It yields the existence of a continuous periodic function whose Fourier series diverges at a given point. In this section, we shall also present a variant of this theorem, which we call the uniform boundedness principle. This theorem is particularly useful for the study of matrix transformations in sequence spaces which are linear metric spaces but not normed linear spaces.

CHAPTER-1

PRELIMINARIES

1. METRIC SPACES

1.1 Definition: Let X be any nonempty set. A metric on X is a mapping $d : X \times X \to R$ which satisfies the following axioms: for all $x, y, z \in X$,

$(i) d(x, y) = 0$

$(ii) d(x, y) = 0 \leftrightarrow x = y$

$(iii) d(x, y) = d(y, x)$

$(iv) d(x, z) \leq d(x, y) + d(y, z)$

The set X together with a metric d on it is called a *metric space* and is denoted by (X, d). We usually omit d and only write X to denote a metric space. The inequality (iv) is called the *triangular inequality*. A function $d : X \times X \to R$ which satisfies (i),(iii) and (iv) is called a *semi-metric* and (X, d) is called a *semi-metric space*

1.2 Remarks: (a) A metric d is always non-negative. For $x, y \in X$ it follows that

$$d(x, y) + d(y, x) \geq d(x, x)$$

i.c,

$$2d(x, y) \geq 0$$

and hence

$$d(x, y) \geq 0$$

. (b) If $x, y, x', y' \in X$ it should be noted that

$$|d(x, y) - d(x', y')| \leq d(x, x') + d(y. y')$$

(c) Every nonempty set X can be made into a metric space in a trivial way. Define $d : X \times X \to R$ by
$d(x, x) = 0$ and $d(x, y) = 1$ for $x \neq y$
It is easy to see that d is a metric on X and this is called the *trivial metric*.

1.3 Examples of metric spaces

(a) For any $n \in N$,

$$R^n = \{x = (x_1, \dots, x_n) : x_i \in R, i = 1, \dots, n\}$$

is a metric space with the metric d defined by

$$d(x, y) = (\sum_{i=1}^{n} (x_i - y_i)^2)^{1/2}, x, y \in R^n$$

(b) For any $n \in N$,

$$C^n = \{x = (x_1, \dots, x_n) : x_i \in C, i = 1, \dots, n\}$$

is a metric space with metric d,

$$d(x, y) = (\sum_{i=1}^{n} (x_i - y_i)^2)^{1/2}, x, y \in C^n$$

(c) Let $C[0, 1]$ denote the set all continuous real valued functions defined on the closed interval [0,1]. Any $f \in C[0, 1]$ is bounded and attains its bounds. If we define

$$d(f, g) = sup\{|f(x) - g(x)| : x \in [0. 1]\},$$

then d is a metric and $C[0, 1]$ is a metric space with this metric. Another metric on $C[0, 1]$ is given by

$$P(f, g) = \int_0^1 |f(x) - g(x)| dx$$

2. TOPOLOGY

2.1 Definition: Let X be any nonempty set. A topology on X is a collection $\mathcal{F}$ of subsets of X which satisfies the following axioms :
(i) $\phi, X \in \mathcal{F}$, (ii) any union of members of $\mathcal{F}$ is a member of $\mathcal{F}$, (iii) the intersection of finite number of members of $\mathcal{F}$ is a member of $\mathcal{F}$.

The set X together with a topology $\mathcal{F}$ is called a *topological space* and is written as $(X, \mathcal{F})$. When there is no chance of confusion we write only X to denote a topological space. The members of the topology $\mathcal{F}$ are called *open sets*.

1.2 Examples of topology

(a) If X is a nonempty set and $\mathcal{F}$ consists of all subsets of X ,then $\mathcal{F}$ is a topology for X and this is called the *discrete topology*.

(b) If X is a nonempty set and $\mathcal{F} = \{\phi, X\}$. then $\mathcal{F}$ is a topology for X and this is called the *indiscrete topology*.

(c) Let $X = R$ and let $\mathcal{F}$ consist of the null set ϕ and all open intervals only. Then $\mathcal{F}$ is not a topology on R as it does not satisfy axiom (ii).

3. OPEN SPHERE OR OPEN BALL

3.1 Definition: Let (X, d) a metric space. Let $a \in X$ and $r > 0$. Then the set

$$B_r(a) = \{x \in X : d(x, a) < r\}$$

is called a *neighbourhood or an open sphere or an open ball* with centre a and radius r.

4. CLOSED SPHERE OR CLOSED BALL

4.1 Definition: Let (X, d) be a metric space. Let $a \in X$ and $r > 0$. Then the set $B_r[a] = \{x \in X : d(x, a) \leq r\}$ is called a *closed sphere or a closed ball* in X with centre a and radius r.

Note: Let X be a topological space. A set $G \subset X$ is said to be open if for each $x \in G$, there exists a $r > 0$ such that $B_r(x) \subset G$.
A set $F \subset X$ is said to be closed if and only if its complement is open.
The interior A^0 of a set A is the union of all open sets contained in A.

5. VECTOR SPACES

5.1 Definition: A vector space or linear space over a field K is a set X with mappings

$$(x, y) \to x + y$$

of $X \times X$ into Y ,called *addition*, and

$$(\lambda, x) \to \lambda x$$

of $K \times K$ into X , called *scalar multiplication*, such that the following axioms are satisfied : for x, y, z in X and λ, μ, in K,
$(i)(x + y) + z = x + (y + z)$

$(ii) x + y = y + x$

(iii) there exists an element $0 \in X$, called the zero vector such that$x + 0 = x$

(iv) for each $x \in X$, there exists an element$(-x) \in X$, called the additive inverse or the negative of x, such that $x + (-x) = 0$

$(v)\lambda(x + y) = \lambda x + \lambda y$

$(vi)(\lambda + \mu)x = \lambda x + \mu x$

$(vii)\lambda(\mu x) = (\lambda\mu)x$

$(viii)1.x = x$

The elements of X are called vectors and those of the field K are called scalars. A vector space X is called real or complex vector space accordind as the field K is R or C.

5.2 Examples of vector spaces:

(a) For any $n \in N$,

$$R^n = \{(x_1, x_2, \ldots, x_n) : x_i \in R, i = 1, 2, \ldots, n\}$$

is a real vector space with respect to addition and scalar multiplication defined as follows :

$$x + y = (x_1 + y_1, \ldots, x_n + y_n)$$

$$\lambda x = (\lambda x_1, \dots, \lambda x_n)$$

where $x = (x_1, x_2, \dots, x_n), y = (y_1, \dots, y_n)$ and $\lambda \in R$

Similarly,

$$C^n = \{(z_1, z_2, \dots, z_n) : z_i \in C, i = 1, 2, \dots, n\}$$

is a complex vector space.

(b) Let S denote the set of all sequences $x = \{x_n\}$ of real(complex) numbers. Then S is a real(complex) vector space under coordinatewise linear operation defined as follows:

$$\{x_n\} + \{y_n\} = \{x_n + y_n\},$$
$$\lambda\{x_n\} = \{\lambda x_n\}$$

(c) Let X be any non-empty set and let $\mathcal{F}(x)$ denote the set of all real valued (complex valued) functions defined on X. Then $\mathcal{F}(x)$ is a real (complex) vector space with respect to pointwise linear operations defined as follows :

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$

$$(\lambda f)x = \lambda f(x),$$

for all $x \in X$, where $f_1, f_2 \in \mathcal{F}(x)$ and λ is real or complex.

6. LINEAR FUNCTIONALS

6.1 Definition : Let X be a complex (real, respectively) vector space. A linear functional on X is a linear map $f : X \to C(f : X \to R$, respectively) that is a function f defined on X whose values are complex (real, respectively) numbers , and such that

$$f(c_1 x_1 + c_2 x_2) = c_1 f(x_1) + c_2 f(x_2)$$

if $c_1, c_2 \in C$ (or if $c_1, c_2 \in R$ respectively).
f is said to be a real or complex linear functional according as it is real or complex valued.

6.2 Example of linear functionals

Let $V^n = C^n$ (or R^n) be the n-dimensional complex (or real) vector space. If $b_1, b_2, \ldots, b_n$ are n fixed complex (or real) numbers and if , for every $x = (e_1, \ldots, e_n) \in V^n$, we define $f(x) = b_1 e_1 + \ldots + b_n e_n$, then f is a linear functional on V^n and $f(e_i) = b_i, i = 1, 2, \ldots, n$ where $e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, 0, \ldots, 1)$ is the basis of V^n.

7. SUBLINEAR FUNCTIONALS

7.1 Definition: Let X be a real vector space. A real valued function p defined on X is called a sublinear functional if

$$p(x + y) \leq p(x) + p(y)$$

and

$$p(ax) = ap(x)$$

for all $x, y \in X$ and all positive real number a.

8. LINEAR METRIC SPACES

8.1 Definition : Let K denote the field R or C. Let X be a vector space over the field K and a topological space. Then X is said to be a linear topological space if the algebraic operations of addition and scalar multiplication are continuous , i.e, the mappings

$$(x, y) \to x + y$$

of $X \times X$ into X and

$$(\lambda, x) \to \lambda x$$

of $K \times X$ into X are continuous. The topologies on $K \times X$ and $X \times X$ are the usual product topologies where K has its usual metric topology defined by the absolute value. If the topolgy of X is given by a metric, then we speak of a linear metric space.

8.2 Definition : Let X be a vector space over the field K. A paranorm on X is a function $g : X \to R$ which satisfies the following axioms: for $x, y \in X$

(i) $g(0) = 0$

(ii) $g(x) = g(-x)$

(iii) $g(x + y) \leq g(x) + g(y)$

(iv)$\lambda \to \lambda_0 x \to x_0$ imply $\lambda x \to \lambda_0 x_0$
where $\lambda, \lambda_0 \in C$ and $x, x_0 \in X$, in other words

$$|\lambda - \lambda_0| \to 0, g(x - x_0) \to 0$$

imply $g(\lambda x - \lambda_0 x_0) \to 0$

(iii) is called the triangle inequality and (iv) is the continuity of scalar multiplication. A paranormed space is a vector space X with a paranorm g and is written as (X, g). We sometimes write only X to denote a paranormed space when there is no chance of confusion. A paranorm is dsaid to be total if

$$g(x) = 0$$

implies $x = 0$

Note: Let X be a linear space over K. A nonempty subset E of X is said to be a subspace of X if $kx + ly \in E$ whenever $x, y \in E$ and $k, l \in K$. If $\phi \neq E \subset X$, then the smallest subspace of X containing E is

$$spanE = \{k_1 x_1 + k_2 x_2 + \ldots + k_n x_n \ : \ x_1, x_2, \ldots, x_n \in E, \ k_1, k_2, \ldots, k_n \in K\}$$

It is called span of E.
A subset E of X is said to be linearly independent if for all $x_1, \ldots, x_n \in E$ and $k_1, \ldots, k_n \in K$, the equation $k_1 x_1 + \ldots + k_n x_n = 0$ implies that $k_1 = k_2 = \ldots = k_n = 0$.
A subset E of X is called a basis for X if $spanE = X$ and E is linearly independent.

8.3 Examples of linear metric spaces

(a) $l(p)$ is a linear metric space with the total paranorm g defined by

$$g(x) = (\sum_k |x_k|^{P_K})^{1/M}$$

where $M = \max$ (1, sup p_k)

(b) $c_0(p)$ is a linear metric space paranormed by g defined by $g(x) = \sup |x_k|^{P_k/M}$

9. HOMEOMORPHISM

9.1 Definition: Let (X,T) and (Y,U) be topological spaces, and let $f : X \to Y$ be a bijection. f is said to be a homeomorphism if f is continuous and its inverse f^{-1} is continuous.

9.2 Examples of homeomorphism:

1) A function $f : X \to Y$ where X and Y are discrete spaces is a homeomorphism if and only if it is a bijection.

2) Let X be a set with two or more elements, and let $p \neq q \in X$. A function $f : (X, T_p) \to (Y, T_q)$ is a homeomorphism if and only if it is a bijection such that $f(p) = q$.

10. PARTIALLY ORDERED SET

10.1 Definition: Let X be any nonempty set. A partial order relation in X is a relation $\leq$ such that for all $x, y, z \in X$ we have
(i) $x \leq x$
(ii) $x \leq y$ and $y \leq x$ imply $x = y$
(iii) $x \leq y$ and $y \leq z$ imply $x \leq z$
A nonempty set X with a partially ordered relation defined on it is called partially ordered set.

11. ZORN'S LEMMA

If X is a nonempty partially ordered set such that every totally ordered subset of X has an upper bound in X. Then X contains a maximal element.

12. LINEAR MAPS

12.1 Definition: Let X and Y be linear spaces over K. A linear map from X to Y is a function $F : X \to Y$ such that

$$F(x_1k_1 + x_2k_2) = k_1F(x_1) + k_2F(x_2)$$

for all $x_1, x_2 \in X$ and $k_1, k_2 \in K$. Two important subspaces are associated with such a map. The subspace

$$R(F) = \{y \in Y : F(x) = y \ for \ some \ x \in X\}$$

of Y is called the range space of F. The subspace

$$Z(F) = \{x \in X : F(x) = 0\}$$

of X is called the zero space of F.

Closure: Let X be a topological space. The closure $\bar{A}$ of a set $A \subset X$ is the intersection of all closed sets that contain A.

Nowhere Dense and Dense: Let X be a topological space. A set $A \subset X$ is said to be nowhere dense if $\bar{A}$ has empty interior. $A \subset X$ is said to be dense in X if $\bar{A} = X$.

13. THE FIRST AND SECOND CATEGORY OF METRIC SPACES

13.1 Definition: Let (X, d) be the metric space. (X, d) is said to be of The First Category if X is equal to the union of a countable collection of nowhere dense subsets of X. (X, d) is said to be of The Second Category if X is not of The First Category.

14. BAIRE CATEGORY THEOREM

Let X be a metric space. Then the intersection of a finite number of dense open subsets of X is dense in X.
If X is complete, then the intersection of a countable number of dense open subsets of X is dense in X.

CHAPTER-2

NORMED LINEAR SPACES AND BOUNDED LINEAR TRANSFORMATIONS

Let X be a real or complex vector space of finite or infinite dimension. Let K be the field R of real numbers or the field C of complex numbers. When the scalar field is not specified, then it is understood that the results are valid for both cases.

2.1 Definition: A norm on X is a real function $\|.\| : X \to R$ defined on X such that for any $x, y \in X$ and for all $\lambda \in K$,
(1)$\|x\| \geq 0$
(2) $\|x + y\| \leq \|x\| + \|y\|$
(3) $\|\lambda x\| = |\lambda|\|x\|$
(4) $\|x\| = 0$ implies $x = 0$

2.2 Remark: The following are the immediate consequences of the definition of a norm:
(a) Take $\lambda = 0$ in property (3), then $\|0\| = 0$; hence property (4) can be stated as follows :

$$\|x\| = 0 \ iff \ x = 0$$

(b) Take $\lambda = -1$ in property (3), then $\| - x\| = \|x\|$, in particular

$$\|y-x\| = \|x-y\|$$

(c) Note that

$$|\|x\| - \|y\|| \leq \|x-y\|$$

It follows from property (2) that

$$\|x\| - \|y\| \leq \|x - y\|$$

Writing x in place of y in the above inequality. we get

$$\|y\| - \|x\| \leq \|y-x\|$$

Thus

$$\|x\| - \|y\| \geq -\|x-y\|$$

Hence,

$$-\|x-y\| \leq \|x\| - \|y\| \leq \|x - y\|$$

as required.

2.3 Definition: A normed linear space is vector space with a norm. We shall abbreviate normed linear space as nls.

2.4 Definition: A *seminorm* is defined by omitting property (4) in the definition of a norm. This concept is important in the theory of topological vector spaces.

A $p-norm$ is defined by replacing the property (3) in the definition of a norm by the following one:

$$\|\lambda x\| = |\lambda|^p \|x\|$$

Note that a $p-norm$ with $p = 1$ is just a norm.

2.5 Examples:

(a) R^n is a nls with the norm defined by

$$\|x\| = (\sum_{j=1}^{n} |x_j|^2)^{1/2}$$

Similarly C^n is a nls with the norm defined in the above manner.

(b) $C(S)$ is a nls with the norm

$$\|f\| = sup\{|f(x)| : x \in S\}$$

(c) A, the set of all complex functions analytic on $\{Z \in C : |z| < 1\}$ and continuous on $\{z \in C : |z| \leq 1\}$ is a nls with a norm

$$\|f\| = \max_{|x|=1} |f(z)|$$

2.6 Proposition: Every nls X is metric space, relative to the natural metric d defined by

$$d(x, y) = \|y-x\|$$

for $x, y \in X$. Furthermore , for any $x, y, z \in X$ and for all $\lambda \in K$, we have

$$\|x\| = d(0, x)$$

as well as

$(a) d(x+z, y+z) = d(x, y)$

$(b) d(\lambda x, \lambda y) = |\lambda| d(x, y)$

Property (a) is called translation invariance

Proof: It can be easily verified that the axioms for a metric hold good. For example,

$$d(x, z) \leq d(x, y) + d(y, z)$$

follows immediately by writing

$$z - x = (y - x) + (z - y)$$

so that

$$\|z - x\| \leq \|y - x\| + \|z - y\|$$

Proof of (a)

$$\begin{aligned} d(x+z, y+z) &= \|(y+z) - (x+z)\| \\ &= \|y - x\| = d(x, y) \end{aligned}$$

Proof of (b)

$$\begin{aligned} d(\lambda x, \lambda y) &= \|\lambda y - \lambda x\| \\ &= \|\lambda(y - x)\| \\ &= |\lambda| \|y - x\| \\ &= |\lambda| d(x, y) \end{aligned}$$

2.12 Proposition: Let X be a normed space. The mappings

1) $(x, y) \in X \times X \to x + y \in X$

2) $(\lambda, x) \in K \times X \to \lambda x \in X$

3) $(x, y) \in X \times X \to d(x, y) = \|y - x\| \in R$

4) $x \in X \to \|x\| \in R$

are all continuous.

Proof : We prove each part seperately

1) Let $(a,b) \in X \times X$ be an arbitrary point , so that $a+b$ is its image. In order to prove that the mapping is continuous at (a,b) , it is sufficient to show that given $\epsilon > 0$, there is a $\delta > 0$ such that for every $x \in B_\delta(a)$ and every $y \in B_\delta(b)$, we have $x+y \in B_\epsilon(a+b)$. In other words, we must show that, given $\epsilon > 0$ there is $\delta > 0$ such that

$$\|(x+y)-(a+b)\| < \epsilon$$

Whenever

$$\|x-a\| < \delta \; and \; \|y-b\| < \delta$$

Taking $\delta = (\frac{\epsilon}{2})$, *we have*

$$\|(x+y)-(a+b)\| \leq \|x-a\| + \|y-b\| < \delta + \delta = \epsilon$$

(2) Let $\alpha \in K$ and $a \in X$ be arbitrary. To prove that the mapping is continuous at (α, a), we must show that, given $\epsilon > 0$, there is $\delta > 0$ such that

$$\|\lambda x - \alpha a\| < \epsilon$$

Whenever

$$|\lambda - a| < \delta \quad \|x - a\| < \delta$$

. We have the identity

$$\lambda x - \alpha a = (\lambda - \alpha)a + \alpha(x-a) + (\lambda - \alpha)(x-a)$$

which yields

$$\|\lambda x - \alpha a\| \leq |\lambda - \alpha|\|a\| + |\alpha|\|x-a\| + |\lambda - \alpha|\|x-a\|$$

Hence, choosing $\delta > 0$ sufficiently small, we get

$$\|\lambda x - \alpha a\| < \delta\|a\| + |\alpha|\delta + \delta_2 \leq \epsilon$$

(3) In this case, the function is the metric of a metric space. It follows from the property of metric spaces that the metric is continuous.

(4) Setting $y = 0$ in case (3), we obtain the function in case (4), which is continuous because the function in case (3) is continuous. This is due to the fact that a function which is continuous as a function of two variables is continuous in each variable separately.

2.13 Proposition: Every nls is a linear metric space with respect to the natural metric defined by $d(x,y) = \|y - x\|$.

Proof: This follows from Proposition 2.6 and parts (1) and (2) of Proposition 2.12

2.14 Corollary : Let $a \in X$ and $\alpha \in K, \alpha \neq 0$. Then the mapping

$$x \in X \rightarrow \alpha x + a \in X$$

is a homeomorphism of X onto itself.

Proof : If $\alpha = 1$, we have the translation

$$x \rightarrow x + a$$

This mapping is continuous by part (1) of Proposition 2.12, letting $y = a$.

If $a = 0$, we have $x \rightarrow \alpha x$, which is continuous, as can be seen by setting $\lambda = \alpha$ in part (2) of Proposition 2.12.

The mapping $x \rightarrow \alpha x + a$, being the composition of the continuous mappings $x \rightarrow \alpha x$ and $x \rightarrow x + a$, is also continuous.

Now,

$$x = \frac{1}{\alpha}y + \frac{-a}{\alpha}$$

if $\quad y = ax + a$.

It follows that $x \rightarrow \alpha x + a$ is a bijection of X with itself, with inverse.

$$y \rightarrow \frac{1}{\alpha}y + \frac{-a}{\alpha}$$

Since the inverse has the same form as the given mapping, it is also continuous. Hence, $x \longrightarrow ax + a$ is homeomorphism.

2.15 Definition : A sequence $\{x_n\}$ in a nls X is a Cauchy sequence if for every $\epsilon > 0$ there exists $n_0 \in N$ such that $\|x_n - x_m\| < \epsilon$ for $n \geq n_0$.

A series $\sum_{n=1}^{\infty} a_n$, $a_n \in X$, is said to be convergent to $x \in X$ if the sequence of partial sums $\{s_n\}$, where $s_n = \sum_{i=1}^{n} a_i$, converges to x, i.e. if for every $\epsilon > 0$ there exists $n_0 \in N$ such that $\|s_n - x\| < \epsilon$ for $n \geq n_0$. A series $\sum_{n=1}^{\infty} a_n$ is said to be absolutely convergent if $\sum_{n=1}^{\infty} \|a_n\|$ is convergent.

Note that every nls X is a metric space and hence every convergent sequence in it is Cauchy, but not conversely.

2.16 Definition : A nls X is said to be complete if every Cauchy sequence in X converges to an element of X. A complete nls is called a Banach space.

2.17 Example : All the nlss given in the Example (2.5): R^n, C^n, $C(S)$, A, I_ρ, I_∞, c, c_0 are Banach spaces.

Example for Cauchy sequence : Let $x_n = \frac{2_n+1}{n}$, then $\{x_n\}$ is Cauchy.

Example for Banach space : One of the simplest examples of a non trivial Banach space is $C(k, k_n)$, the space of continuous maps from a compact space k to k_n, with $k = C$ or R.

BOUNDED LINEAR TRANSFORMATIONS

We shall first introduce the concept of bounded linear transformations.

2.18 Definition : Let k denote the real field R or the complex field C and let X and Y be normed linear spaces over k. A linear transformation T from X and Y is said to be a bounded linear transformation if there exists a constant $M > 0$ such that,

$$\|Tx\| \leq M\|x\| \; for \; every \; x \in X$$

In the above inequality $\|x\|$ is the norm of x in X and $\|Tx\|$ is the norm of Tx in Y. It will frequently happen that several norms occur together, but we will use the same symbol for all the norms. Seeing the nature of the discussion the reader can clearly distinguish between different norms.

If T is a bounded linear transformation, the norm of T is defined by

$$\|T\| = sup\{\frac{\|Tx\|}{\|x\|} : x \in X, x \neq 0\}$$

It may be noted that we can restrict ourselves only to $x \in X$ with $\|x\| = 1$ without changing the supremum, since for $\alpha \in K$,

$$\|T(\alpha x)\| = \|\alpha Tx\| = |\alpha|\|Tx\|$$

Therefore, the norm of T can be defined by

$$\|T\| = sup\{\|Tx\| : x \in X, \|x\| = 1\}$$

Also, it may be noted that

$$\|T\| = inf\{M : M > 0, \|Tx\| \leq M\|x\|\} \ for \ all \ x \in X$$

In other words,

$\|Tx\| \leq \|T\|\|x\|$ for all $x \in X$

If T is a bounded linear transformation from a normed linear space X into itself, then we call T a bounded linear operator. A bounded linear transformation from X into the field K is called bounded linear functional; it is called a real or complex bounded linear functional according as K is the real field R or the complex field C.

2.19 Examples: (a) The identity operator $I : X \longrightarrow X$ on a normed linear space $X \neq 0$ is a bounded linear operator with a norm $\|I\| = 1$.

(b) The zero transformation $0 : X \longrightarrow Y$ on a normed linear space X is a bounded linear transformation and has the norm $\|0\| = 0$.

(c) The norm $\|.\| : X \longrightarrow R$ on a linear space X is not a linear functional; it is a sublinear functional.

CHAPTER - 3

The HAHN-BANACH THEOREM

In this chapter we shall prove the famous Hahn banach theorem. This is one of the most fundamental theorems in functional analysis and is due to Hahn and Banach . It yields the existence of nontrivial continuous linear functionals on a normed linear space ,a basic result necessary for the development of a large portion of functional analysis .Moreover , it is an indispensible tool in the proofs of many important theorems in analysis.

3.1 The Hahn-Banach Theorem: Let E be a real linear space and let M be a linear subspace of E. Suppose p is a sublinear functional defined on E and f a linear functional defined on M such that $f(x) \leq p(x)$ for every $x \in M$, then there is a linear functional g defined on E such that g is an extension of f (i.e $g(x) = f(x)$ for all $x \in M$) and $g(x) \leq p(x)$ for all $x \in E$.

Proof: Let F denote the set of all real functions h, such that h is linear, *dom* h is a linear subspace of E ,h is an extension of f and $h(x) \leq p(x)$ for all $x \in$ *dom* h.Since $f \in F$,$F \neq \phi$. We write $f \subset h$ to denote that h is an extension of f. Notice that F is a partially ordered set with respect to partial order $\subseteq$. Let C be any chain in F and let h =∪C . Then $h \in F$, therefore from zorn's lemma it follows that F has a maximal element ,say g. We complete the proof by showing that dom $g = E$. Assume that this is false i.e ,let *dom* $g = G \subset E$, let y be any element in $E \cap G^c$, define

$$H = \{x + \alpha y : x \epsilon G. \alpha \epsilon R\}$$

Clearly H is a linear subspace of E and $G \subset H$,let c be a fixed , but arbitrary , real number,define h on H by

$$h(x + \alpha y) = g(x) + \alpha c$$

Now if $x_1 + \alpha_1 y = x_2 + \alpha_2 y$, where $x_1, x_2 \in G$ and $\alpha_1, \alpha_2 \in R$,then $(\alpha_1 - \alpha_2)y = x_2 - x_1 \in G$ so that $\alpha_1 = \alpha_2$ and $x_1 = x_2$). Hence h is well defined .Clearly h is linear and $g \subset h$. We now claim that a 'c' can be selected so that $h(x) \leq p(x)$ for all $x \in H$: then we have $h \in F$, which contradicts the maximality of g and completes the proof of the theorem, therefore,we need only to establish that c can be so selected , to justify our claim, thus our requirement is that ,

$$g(x) + \alpha c = h(x + \alpha y) \geq p(x = \alpha y) \ for \ all \ x \in G, \alpha \in R.$$

Since g is linear and p sublinear ,it is equivalent to requiring that

$$g(\frac{x}{\alpha}) + c \leq p(x\alpha + y) \ for \ x \in G \ and \ \alpha > 0$$

and

$$g(\frac{x}{\alpha}) + c \leq -p(-x\alpha - y) \ for \ x \in G \ and \ \alpha < 0$$

.

Therefore it suffices to have

$$g(u) - p(u - y) \leq c \leq -g(v) + p(v + y) \ for \ u, v \in G$$

but we do have

$$g(u) + g(v) = g(u + v) \leq p(u + v) \leq p(u - y) + p(v + y)$$

for all $u, v \in G$.Write

$$a = sup g(u) - p(u - y) : u \in G$$

$$b = inf - g(v) + p(v + y) : v \in G$$

It is clear that $a \leq b$. Now any real number c such that $a \leq c \leq b$ satisfies our requirements.

3.2 Theorem : Let E be a real normed linear space and let M be a linear subspace of E. If $f \in M^*$, then there is a $g \in E^*$ such that $f \subset g$ and$\|g\| = \|f\|$.

Proof: Define p on E

$$p(x) = \|f\|\|x\|$$

Then clearly, p is a sublinear functions on E and $f(x) \leq |f(x)| \leq p(x)$ for all $x \in M$. therefore , it follows from Theorem(3.1) that there exist a linear functional g on E such that $f \subset g$ and $g(x) \leq p(x)$ for all $x \in E$, clearly $g \in E^*$ and $\|g\| \leq \|f\|$. Also

$$\begin{aligned} \|g\| &= sup\{|g(x)| : x\epsilon E, \|x\| = 1\} \\ &\geq sup\{|g(x)| : x\epsilon M, \|x\| = 1\} \\ &= sup\{|f(x)| : x\epsilon M, \|x\| = 1\} \\ &= \|f\| \end{aligned}$$

Therefore $\|g\|=\|f\|$ and this completes the proof.

3.3 Theorem : Let E be complex linear space and let M be a linear subspace of E, suppose q is a seminorm on E and f is a linear functional defined on M such that ,

$$|f(x)| \leq q(x) \ for \ all \ x \in M$$

Then there is a linear functional g on E such that $f \subset g$ and

$$|g(x)| \leq q(x) \ for \ all \ x \in E$$

Proof: Let for each $x \in M$,write $f(x) = f_1(x) + if_2(x)$. An easy computation shows that f_1 and f_2 are real linear functional on M, it is also obvious that for $j = 1, 2$

$$|f_1(x)| \leq |f(x)| \leq q(x)$$

for all $x \in M$. Now we regard E and M as real linear spaces and apply theorem(3.1) to obtain a real linear functional g_1 on E such that $f_1 \subset g_1$ and

$$|g_1(x)| \leq q(x)$$

for all $x \in E$
Now define g on E by the rule

$$g(x) = g_1(x) - ig_1(ix)$$

It is easy to see that g is a complex linear functional on E . Further , for $x \in M$,

$$\begin{aligned} g_1(ix) + if_2(ix) &= f_1(ix) + if_2(ix) \\ &= f(ix) = if(x) \\ &= -f_2(x) + if_1(x) \\ &= -f_2(x) + ig_1(x) \end{aligned}$$

so that $g_1(ix) = f_1(x)$ and therefore

$$g(x) = g_1(x) - ig_1(ix) = f_1(x) + if_2(x) = f(x)$$

thus $f \subset g$ it remains only to show that

$$|g(x)| \leq q(x) \ for \ all \ x \in E$$

Let $x \in E$ be arbitrary . Write $g(x) = rexp(i\theta), r \geq 0, \theta \in R$.

$$|g(x)| = r = exp(-i\theta)g(x) = g(exp(-i\theta)x)$$

$$= g_1(exp(-i\theta)x \geq q(exp(-i\theta)x)$$

$$= |exp(-i\theta)|q(x) = q(x)$$

This completes the proof.

3.4 Theorem: Let E be a complex normed linear space and let M be a linear subspace of E. If $f \in M^*$, then there exist $g \in E^*$ such that $f \subset g$ and $\|g\|= \|f\|$.

This can be obtained from 3.2 and 3.3.
We now present some results which are applications of Hahn banach theorem.

3.5 Theorem : Let E be a normed linar space over K and let S be a linear subspace of E. Suppose that $z \in E$ and dist $(z, s)= d \geq 0$, then there exists $g \in E^*$such that $g(S)$=0,$g(z)$,$g(z) = d$ and $\|g\|$=1.

Proof: Let

$$M = x + \alpha z : x \in S, \alpha \in K$$

Then clearly M is a linear subspace of E ,define f on M by

$$f(x + \alpha z) = \alpha d$$

Clearly f is a well defined linear functional on M .Also $f(S)$=0 and $f(z) = d$ .Further

$$\|f\| = sup\{\frac{|f(x + \alpha z)|}{\|x + \alpha z\|} : x + \alpha z \in M, \|x + \alpha z\| \neq 0\}$$

$$= sup\{\frac{|\alpha d|}{\|x+\alpha z\|} : x+\alpha z \in M, \|x+\alpha z\| \neq 0\}$$

$$= sup\{\frac{d}{\|-y+z\| : y \in S}\}$$

$$= \frac{d}{d} = 1$$

Thus $f \in M^*$, now by applying theorem 3.2 or 3.4 as the case may be , we obtain a functional $g \in E^*$ which satisfies the requirement of the theorem.

3.6 Corollary: Let E be a normed linear space and let z be a non zero vector in E . Then there exists functional $g \in E^*$ such that $g(z) = \|z\|$ and $\|g\| = 1$.

Proof: Take S =0 in theorem 3.5.

This answers the question raised in the previous section that if E is a nonzero normed linear space , then there exists a non zero element in E^*.

3.7 Discussion:

If E is a normed linear space ,then there exists a natural mapping from E into E^{**} . Each element $x \in E$ gives rise to a functional F_x in E^{**} defined by

$$F_x(f) = f(x)$$

For $f \in E^*$. We denote the natural mapping $x \to F_x$ from E into E^{**} by π. A simple computation shows that F_x is a linear functional on E^*. Further more ,

$$\|F_x\| = sup|F_x(f)| : \|f\| \leq 1$$

$$= sup|f(x)| : \|f\| \leq 1$$

$$\leq sup\|f\|\|x\| : \|f\| \leq 1$$

$$\leq \|x\|$$

Thus $F_x \in E^{**}$ and π is well defined. Also the mapping π is linear and since

$$\|\pi(x)\| = \|F_x\| \leq \|x\|$$

it follows that π is a bounded linear transformation from E into E^{**} with $\|\pi\| \leq 1$.

CHAPTER - 4

CLOSED GRAPH THEOREM AND OPEN MAPPING THEOREM

Before stating our main theorem, we prove a crucial preliminary result

4.1 Lemma: Let X be a linear space over K. Consider subsets U and V of X, andv $k \in K$ such that $U \subset V + kU$. Then for every $x \in U$, there is a sequence (v_n) in V such that

$$x - (v_1 + kv_2 + \ldots + k^{n-1}v_n) \in k^n U, \qquad n = 1, 2, \ldots$$

Proof: Let $x \in U$. Since $U \subset V + kU$, there is some $v_1 \in V$ such that $x - v_1$ is in kU. Let $n \geq 1$ and assume that we have found $v_1, v_2, \ldots, v_n$ in V as stated in the lemma. Then $x = v_1 + kv_2 + \ldots + k^{n-1}v_n + k^n u$ for some $u \in U$. Since $u = v_{n+1} + ku_0$ for some $v_{n+1} \in V$ and $u_0 \in U$, we see that

$$x - (v_1 + kv_2 + \ldots + k^n v_{n+1} U.$$

Thus we inductively obtain a sequence (v_n) in V.

4.2 Closed Graph Theorem: Let X and Y be Banach spaces and $F : X \to Y$ be a closed linear map. Then F is continuous.

Proof: It is enough to prove that F is bounded on some neighbourhood of 0. For each positive integer N, let

$$V_n = \{x \in X : \|F(x)\| \leq n\}$$

We prove that some V_n contains a neighbourhood of 0 in X. Now

$$X = \bigcup_{n=1}^{\infty} V_n = \bigcup_{n=1}^{\infty} V'_n$$

where V'_n denotes the closure of the set V_n in X. Hence

$$\bigcap_{n=1}^{\infty} (V'_n)^c = \phi$$

where $(V'_n)^c$ denotes the complement of the set V'_n in X. Since X is a Banach space, one of the open sets $(V'_n)^c$ must not be dense in X.

Hence we find a positive integer p , some $x_0 \in X$ and $\delta > 0$ such that $U(x_0, \delta) \subset V'_p$. We shall show that $U(0, \delta) \subset V_{4p}$.

First note that $U(0, \delta) \subset V'_{2p}$. For if, $x_0 \in X$ with $\|x\| < \delta$, then $x + x_0 \in U(x_0, \delta) \subset V'_p$. Also $x_0 \in V'_p$. If (v_n) and (w_n) are sequences in V_p such that $v_n \to x + x_0$ and $w_n \to x_0$, then $v_n - w_n \to x$, where $v_n - w_n \to V_{2p}$ since

$$\|F(v_n - w_n)\| \leq \|F(v_n)\| + \|F(w_n)\| \leq 2p$$

Thus $x \in V'_{2p}$. In particular, for every $x \in U(0, \delta)$, there is some x_1 in V_{2p} such that $\|x - x_1\| < \delta/2$. Hence

$$U(0, \delta) \subset V_{2p} + (1/2)U(0, \delta)$$

Consider $x \in U(0, \delta)$. Letting $U = U(0, \delta), V = V_{2p}$ and $k = 1/2$ in 4.1, we see that there is a sequence (v_n) in V_{2p} such that

$$x - (v_1 + v_2/2 + \ldots + v_n/2^{n-1}) \in (1/2^n)U(0, \delta)$$

for each $n = 1, 2, \ldots$. Let

$$w_n = v_1 + v_2/2 + \ldots + v_n/2^{n-1}, n = 1, 2, \ldots$$

Since $\|x - w_n\| < \delta/2^n$, it follows that $w_n \to x$ in X. Also for all $n > m$, we have

$$\|F(w_n) - F(w_m)\| = \|F(\sum_{j=m+1}^{n} v_j/2^{j-1})\| \leq \sum_{j=m+1}^{n} \|F(v_j)\|/2^{j-1} \leq 4p/2^m$$

Hence $(F(w_n))$ is a Cauchy sequence in Y. As Y is a Banach space, $(F(w_n))$ converges in Y, and as F is a closed map, we see that $F(w_n) \to F(x)$ in Y. If we let $m = 0$ and $w_0 = 0$, then $\|F(w_n)\| \leq 4p$ for all $n \geq 1$ by the calculation given above. Hence

$$\|F(x)\| = \lim_{n\to\infty} \|F(w_n)\| \leq 4p$$

. Since $x \in U(0, \delta)$ is arbitrary, we see that $U(0, \delta) \subset V_{4p}$. Thus the Linear map F is bounded on the neighbourhood $U(0, \delta)$ of 0.

We consider an interesting consequences of the closed graph theorem. A linear map P from a linear space X to itself is called a projection if $P^2 = P$. If P is a projection ,then so is $I - P$ and $R(P) = Z(I - P), Z(P) = R(I - P)$. It follows that

$$X = R(P) = Z(P) \textit{ and } R(P) \cap Z(P) = 0$$

For every projection P defined on X . Conversely ,if Y and Z are subspaces of X such that $X = Y + Z$ and $Y \cap Z = 0$, then for every $x \in X$ there are uniques $y \in Y$ and $z \in Z$ such that $x = y + z$,so that the linear map is given by $P(x) = y$ is a projection .It is called the projection onto Y on Z.

The closedness and the continuity of a projection can be determined by the closedness of its range space and zero space .

4.3 Theorem: Let X be normed space and $P : X \rightarrow X$ be a projection .Then P is a closed map if and only if the subsequence $R(P)$ and $Z(P)$ are closed in X . In this case P is in fact continuous ,if X is a Banach space.

Proof: Let P be a closed , $y_n \in R(P)$, then $y_n \rightarrow y$,$z_n \rightarrow z$ in X . then $P(x_n) - y_n \rightarrow y$, $P(z_n) - 0 \rightarrow 0$ in X so that $P(y) = y$ and $P(z) = 0$. Thus $y \in R(P) and z \in Z(P)$, showing that the subsequence of the $R(P)$ and $Z(P)$ are closed in X .

Conversely ,Let $R(P)$ and $Z(P)$ be closed in $X, x_n \rightarrow x$ and $P(x_n) \rightarrow y$ in X . Since $R(P)$ is closed and $P(x_n) \in R(P)$ we see that $y \in R(P)$. Also since $Z(P)$ is closed and $x_n - P(x_n) \in Z(P)$ we see that $x - y$ is in $Z(P)$. Thus $x = y + z$ with $y \in R(P)$ and $z = x - y \in Z(P)$. Hence $p(x) = y$ showing that P is a closed map .
If X is a Banach space and the subsequence $R(P) and Z(P)$ are closed then by the closed graph theorem ,the closed map P is in fact ,continuous .

We remark that if X is a linear space and Y is a subspace of X , then there exists a projection P defined on X such that $R(P) = Y$. For it ,if a basis y for Y is a extended to a basis $y_s \cup z_t$ for X and we let $Z = span z_t$ then clearly $X = Y + Z$ and $Y \cap z = 0$ showing that the projection is onto Y along Z . Now if X is a normed space and Y is a closed subspace of X . Does there exists a closed projection defined on X such that $R(P) = Y$? Such a projection exists if and only if there is a closed subspace of Z of X such that $X = Y + Z$ and $Y \cap 0$. In this case Z is called a closed complement of Y in X .It is known as that c_0 has no closed complement in l_∞ and that $C([0,1])$ has no closed complement in $B([0,1])$ on the other hand , if Y is a finite dimensional subspace of a normed space of X , then there exists a closed complement of Y in X .

We pass on to consider the closely related result known as the open mapping theorem. A map F from a metric space X to a metric space Y is said to be open if for every open set$E$ in X ,its image $F(E)$ is open in Y .Note that a map F is continous if and only if for every open set E in Y ,its inverse image $F^{-1}(E)$ is open in E .

4.4 Theorem : Let X and Y be a normed spaces and $F : X \to y$ be linear .Then F is an open map if and only if there exists some $\gamma > 0$ such that for every $y \in Y$, there is some $x \in X$ with $F(x) = y \ and \ ||x|| \leq \gamma||y||$. In particular if a linear map is open , then there is surjective .

Proof: Let Fbe a open map. Since $U_x(0,1)$ is open in X the set $F(U_x(0,1))$ is open in Y . As $0 = F(0) \in F(u_x(0,1))$, here exists some $\gamma > 0$ .Hence there is some $x_1 \in U_x(0,1)$ such that $F(x_1) = \frac{\delta_y}{||Y||}$. Letting $x = \frac{||y||_x 1}{\delta}$ We see that $F(x) = y$ and $||x||$¡$\frac{||y||}{\delta}$.

Conversely ,assume that for every $y \in Y$ there is some $x \in X$ with $F(x) = y$and $||x|| \leq \gamma||y||$ for some fixed $\gamma > 0$ .Consider an open set E in X and $x_0 \in E$. Then $U_x(x_0, \gamma) \cap E$ for some $\delta > 0$. Let $y \in Y$ with $||y - F(x_0)|| < \frac{\delta}{\gamma}$.By hypothesis , there is some $x \in X$ such that $F(x) = y - F(x_0)$ and $||x|| \leq \delta||y - F(x_0)||$. Then $y = F(X) + F(x_0) = F(x + x_0)$ where $x+x_0 \in U_x(x_0, \delta) \in E$, since $||x|| < \delta$.Thus $U_Y(F(x_0), \frac{\delta}{\gamma} \in F(E)$.Hence $F(E)$ is an open set in Y. We conclude that F is an open map.

4.5 Theorem : Let X and Y be a normed spaces.

(a) If Z is a closed subspace of X ,then the quotients map Q from X to $\frac{x}{y}$ is continous and open .
(b) Let $F : X \to y$ be a linear map such that the subspaces $Z(F)$ is closed in X. Define $F : \frac{x}{Z(F)} \to Y$ by $F(x + Z(F)) = F(x)$ for $X \in X$. Then F is an open map if and only if F is an open map.

Proof: (a) The map Q is continuous because $|||Q(x)||| = |||x+Z||| \leq ||x||$ for all $x \in X$. To show that the linear map Q is an open. We use the results given in 4.4 . Consider any $\epsilon > 0$ Let $x + Z \in \frac{X}{Z}$. Then

$$inf||x + z|| : z \in Z = |||x + Z||| < (1 + \epsilon)|||x + Z|||$$

So that there is some $z_0 \in Z$ with $||x + z_0|| < (1 + \epsilon)|||x + Z|||$. Since $Q(x + z_0) = x + Z$ we can let$\gamma = 1 + \epsilon$ in 4.4 and conclude that Q is an open map.

(b) Since $Z(F)$ is a closed subspaces of $X, \frac{X}{Z(F)}$ is a normed space in the quotient norm. Let $Q : X \to \frac{X}{Z(F)}$ be the quotient map. For $E \in X$,We have $F(E) + F(Q(E))$. As the map Q is open by part (a), it follows that F is open whenever F is open . For $E \in |frac x z(F)$ we have $F(E) = F(Q-^1(E))$,since the map Q is surjective . As the map Q is continuous by part (a) it follows that F is open whenever F is open.

4.6 Open mapping theorem: Let X and Y be the banach space and $F : X \to Y$ be a linear map which is closed and surjective .Then F is continuous and open.

Proof: By the closed graph theorem, F is continuous.Also $Z(F)$ is closed in X and that the map $F' : \frac{X}{Z(F)} \to Y$ is continuous , where $F'(x + Z(F))$ $= F(x), x \in X$. In particular F' is a closed map . Clearly it is injective. Also since the map F is surjective ,so is F' . Thus F' is a bijective closed linear map . Hence it is an inverse map $G' : Y \to \frac{X}{Z(F)}$ is closed and linear . As Y and $\frac{X}{Z(F)}$ are banach spaces , the closed graph theorem shows that G' is continuous, that is F' is open. F is an open map.

We consider an application of the open mapping theorem to the solutions of operator equations. Let X and Y be banach spaces and $F \in BL(X, Y)$. Suppose that for every $y \in Y$, the operator equation

$$F(x) = y$$

has a solution in X, that is the map F is surjective . Then there exists some $\gamma > 0$ such that for every $y \in Y$, the above mentioned operator equation has ,in fact , a solution x in X whose norm is at most $\gamma\|y\|$ .This estimate on the norm of a solution in terms of the norm of the so called free term of the equation is important in many situations.
One such situation occurs when although a unique solution is known to exist in X for every y in Y one is able to find the such a solution only for y belonging to a specified dense subset E of Y. If $y \in Y$ but $y \notin E$, then one may find a sequence (y_n) in E such

that $y_n \to y$ and a sequence (x_n) in X such that $F(x_n) = y_n, n = 1, 2. \ldots$ A natural question arises whether the sequence (x_n) will converge to the unique element x of X such that $F(x) = y$. The answer is in the affirmative ,for

$$F(x - x_n) = F(x) - F(x_n) = y - y_n$$

so that $\|x - x_n\| \leq \gamma\|y - y_n\| \to 0$ as $n \to \infty$. Hence x_n can be called an approximate solution of $F(x) = y$. This also shows that the solutions $x \in X$ corresponding to $y \in Y$ depends continuously on y. We have , therefore established the validity of the perturbation technique used in the theory of operators equations .It consists of changing the free term a little bit and admitting a small change in the solution .

Let us describe a concrete case. Consider an m^th order nonhomogeneous linear ordinary differential equation with variable coefficients:

$$a_m(t)x^m(t) + ... + a_0(t)x(t) = y(t), \quad t \in [a, b]$$

where each $a_j \in C([a, b])$ and $a_m(t) \neq 0$ for every $t \in [a, b]$. Also consider the initial conditions

$$x(a) = x'(a) = ... = x^k(a) = 0$$

where$0 \leq k \leq m - 1$ it is well known that for every $y \in C([a, b])$, there is a solution of the above mentioned diferential equation which satisfies the initial conditions and such a solution is unique if $k = m - 1$.

Suppose that $k = m - 1$, let $Y = C([a, b])$ and

$$X = \{x \in C^m([a, b]) : x(a) = ... = x^{m-1}(a) = 0\}$$

for $x \in X$ let

$$F(x) = a_m x^m + ... + a_0 x$$

then $F : X \to Y$ is linear and bijective .Also if we consider the sup norm $\|\|_\infty$on Y and the norm given by

$$\|x\| = \|x\|_\infty + ... + \|x^m\|_\infty$$

on X, then X and Y are banach spaces and $F \in BL(X, Y)$ since

$$\|F(x)\|_\infty \leq (\|a_m\|_\infty + ... + \|a_0\|_\infty)\|x\|, x \in X$$

Let E denote the set of all polynomials on$[a, b]$ and suppose that for every

$p \in E$ we are able to find the unique element x of X such that $F(x) = p$.

For convenience ,let $a = 0$ and $b = 1$. Consider a continuous function y on $[0, 1]$ which is not a polynomial. For $n = 1, 2, 3, ...$, let

$$p_n(t) = \sum_{k=0}^{n} y(\frac{k}{n})(\frac{n}{k})t^k(1-t)^{n-k}, \quad t \in [0, 1]$$

As we know ,$\|p_n - y\|_\infty \to \infty$ as $n \to \infty$. For each n find $(x_n) \in X$ such that

$$a_m x_n^m + ... + a_0 x_n = p_n$$

By what we have seen before ,it follows that the sequence (x_n) converges in X to the unique element x in X such that

$$a_m x^m + ... + a_0 x = y$$

.

In other words, the approximate solution (x_n) of the initial value problem converges to the exact solution X uniformly on $[a, b]$,and so does j^{th} derivative x_n^j to x^j,$j = 1, \ldots, m$ as $n \to \infty$

Similarly ,consider the multi point boundary value problem

$$a_m(t)x^m(t) + \ldots + a_0(t)x(t) = y(t), \quad x(t_1) = \ldots = x(t_m) = 0$$

where $a = t_1 < \ldots < t_m = b$ and $a_0, \ldots, a_m$ are continuous functions on $[a, b]$. Assume that for every $y \in C([a, b])$, there is a unique solution $x \in C([a, b])$. Approximations to such a solution can be found as in the case of an initial value problem . The theory of numerical solutions of differential problems is based on this tecnique.

4.7 Examples: We give several examples to show that the closed graph theorem and the open mapping theorem may not hold if the normed spaces X and/or Y are not Banach spaces.

(a) Let $X = Y = c_0 0$ with the norm $\|\|_\infty$. We have seen that $c_0 0$ is not a banach space . For $x = (x(1), x(2), ...)$ in X, let

$$F(x)(j) = jx(j), j = 1, 2, ...$$

Then F is a linear mapping from X to Y , also F is closed . To see this ,let

$(x_n) \to x$ in X and $F(x_n) \to y$ in Y, then for every fixed $j = 1, 2, ..., x_n(j) \to x(j)$ as $n \to \infty$,so that $jx_n(j) = F(x_n)(j)$ converges to $jx(j)$ as well as to $y(j)$.thus $y(j) = jx(j) = F(x_n)(j)$ converges to $jx(j)$ as well as to $y(j)$, thus $y(j) = jx(j)$ for all $j = 1, 2, ..$, that is ,$Y = F(x)$. However F is not continuous ,because if we let $x_n = (0, ...0, 1, 0, 0..)$ where 1 occurs only in the nth entry , then $\|X_n\|_\infty = 1$ but $\|F(x_n)\|_\infty = n \to \infty$.

$$F^{-1}(y)(j) = \frac{y(j)}{j}, y \in X, j = 1, 2, ...$$

is continuous since $\|F^{-1}(y)\|_\infty \leq \|y\|_\infty$ for all $y \in Y$. Thus f^{-1} is a closed linear map ,which is also surjective. But it is not an open map since F is not continuous.

Next ,consider $P : X \to X$ defined by

$$P(x)(2j-1) = x(2j-1) + jx(2j)$$

,and $P(x)(2j) = 0$
for$x \in X$ and $j = 1, 2, ...4$.Then P is a linear and $P^2 = P$. Also,$R(P) = \{x \in x : x(2j) = 0$ for $j = 1, 2, ..\}$and $Z(P) = x \in X : x(2j-1) + jx(2j) = 0$for $j = 1, 2, ...$ are both closed subspaces of X . Hence P is a closed map .However ,P is not continuous since if we let $x_n = (1, ...1, 0, 0. ...)$ where 1 occurs in the first 2n entries ,then $\|x_n\|_\infty = 1$,but $\|P(x_n)\|_\infty = n + 1$, which tends to the infinity as $n \to \infty$.

(b) Let $X = C^1([a, b])$ and $Y = C([a, b])$, both with the sup norm $\|\|_\infty$,then Y is a banach space , since X is a proper dense subspace of Y, X is not a banach space for $x \in X$,let $F(X) = x'$, the derivative of x,then $F : X \to Y$ is clearly linear. Also ,it is a closed map. To see this ,let $x_n \to x$ in X and $x'_n \to y$ in Y . Since convergence in the sup norm is nothing but uniform convergence ,an elementary theorem in analysis shows that x is differentiable on $[a, b]$ and $x' = y$, that is $y = F(x)$. However ,F is not continuous ,because if we let $x_n(t) = [(t-a)/(b-a)]^n, t \in [a, b]$, *then* $\|x_n\|_\infty \leq 1$ but $\|F(x_n)\|_\infty = n \to \infty$. This is prime example of a closed linear map which is not continuous . Many differential operators fall in this category.

(c) Let $X = C^1([a, b])$ with a norm given by $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and $Y = C^1([a, b])$ with the sup norm .Then it is can be seen that X is a banach space but Y is not. For $x \in X$, let $F(x) = x$, then $F : X \to Y$ is clearly linear. Also it is continuous since

$$\|F(x)\|_\infty = \|x\|_\infty \leq \|x\|_\infty + \|x'\|_\infty = \|x\|$$

for all $x \in X$ .However F is not an open map since the inverse map F^{-1} : $Y \to x$ discontinuous.

CHAPTER 5

BANACH-STEINHAUS THEOREM

In this section we shall present the famous Banach-Steinhaus theorem which is one of the most celebrated results in the theory of Banach spaces. This theorem has various important applications. It yields the existence of a continuous periodic function whose Fourier series diverges at a given point. In this section, we shall also present a variant of this theorem, which we call the uniform boundedness principle. This theorem is particularly useful for the study of matrix transformations in sequence spaces which are linear metric spaces but not normed linear spaces.

5.1 Banach-Steinhaus theorem: Let $\{T_1\}$ be a nonvoid family of bounded linear transformations from a Banach space X into a normed linear space Y. If sup $\|T_1 x\| < \infty$ for each $x \in X$, then sup $\|T_1\| < \infty$.

Proof: For each positive integer n, define

$$F_n = \{x : x \in X \ and \ \|T_1 x\| \leq n \ for \ all \ i\}$$

Fn is clearly a closed subset of X and $X = \bigcup_{n=1}^{\infty} F_n$. Since X is complete, by the Baire category theorem, one of the F_n's, say F_{n_0}, has a nonempty interior. Thus F_{n_0} contains a closed sphere S_0 with centre x_0 and radius $r_0 > 0$. Hence each vector in each of the sets $T_i(S_0)$ has norm less than or equal to n_0, i.e., $\|T_1 x\| \leq n_0$ for $x \in S_0$ and for $i = 1, 2, \ldots$. We write this as $\|T_1(S)\| \leq n_0$ for$i = 1, 2, \ldots$. Since S_0 is a closed sphere with centre x_0 and radius r_0. $\frac{S_0 - x_0}{r_0}$ is the closed unit sphere S. Since $x \in S_0$, it is evident that

$$\|T_i(S_0 - x_0)\| \leq 2n_0$$

This yields $\|T_1(S)\| \leq \frac{2n_0}{r_0}$. Hence $\|T_1\| \leq \frac{2n_0}{r_0}$ for every i and this completes the proof.

5.2 Remark Note that Banach-Steinhaus theorem need not be true if X is not complete. To observe this, let

$$X = \{(x_1, x_2, \ldots, x_n, 0, 0, \ldots) : x_i \in R, i = 1, 2, \ldots n\}$$

Then X$\subset l_2$. Define $T_n : X \to l_2$ linearly by

$$T_n(e_1) = \begin{cases} 0 & ,i \neq n \\ ne_n & ,i = n \end{cases}$$

where e_1 is a sequence with 1 in the i^{th} place and zeros elsewhere.

Then for $x = \sum_{i=1}^{k} x_i e_i$, $T_n(x) = 0$ for $n > k$. Hence $sup_{i \leq n < \infty} \|T_n x\|$ is finite.
But

$$\|T_n\| = n$$

and

$$sup_{i \leq n < \infty} \|T_n\| = \infty.$$

5.3 Uniform boundedness principle:

Let $\{P_1\}$ be a collection of real lower semi-continuous functions defined on a second-category metric space X. If

$$p_i(x) \leq M(x) < \infty \; for \; all \; x \in X \; and \; all \; i$$

then there exists a sphere S in X and a constant M such that

$p_i(x) \leq M$ for all $x \in X$ and all i

Proof: For each p_1 and each positive integer m define

$$E(m, p_i) = \{x, p_i(x) \leq m\}$$

Clearly$E(m, p_i)$ is closed and $E_m = \bigcap_i E(m, p_i)$ being an intersection of closed sets is closed. Now $X = \bigcup_m E_m$. For if $x \in X$, then $p_i(x) \leq M(x)$ for all i and so there is an integer $m(x)$ such that $p_i(x) \leq m(x)$ for all i. This implies that $x \in E_{m(x)}$.

This proves that $X = \bigcup_m E_m$ and since X is of the second category, this implies that at least one of the sets E_m, say E_M, is not nowhere dense (for the first category). Since E_M is not nowhere dense we have that E_M contains some sphere S and the fact that E_M is closed implies that $S \subset E_M = E_M$. Finally x $\in$ S implies $x \in E_M$, which implies $p_i(x) \leq M$ for all i, and this completes the proof.

CONCLUSION

This Project contained five chapters. First chapter dealt with the preliminaries which included some basic definitions and examples of metric spaces, topology,open and closed ball,vector spaces, linear functionals. linear metric space etc. Second Chapter dealt with Normed Linear Spaces and Bounded linear transformations. It included the definition of norm, normed linear space,seminorm,p-norm, Cauchy sequence, Banach space, Bounded Linear Transformation, some propositions and one corollary.Third Chapter dealt with the famous Hahn-Banach Theorem. This is one of the most fundamental theorems in functional analysis and is due to Hahn and Banach. It yielded the existence of non-trivial continuous linear functionals on a normed linear space, a basic result necessary for the development of a large portion of functional analysis.This chapter also included some more theorems. Fourth chapter dealt with Closed Graph Theorem and Open Mapping theorem. The closed graph theorem is an important result in functional analysis that guarantees that a closed linear operator is continuous under certain conditions.The open mapping theorem asserts that certain continuous linear transformations between Banach spaces map open sets into open sets. The chapter also included some more theorems and examples.The last chapter dealt Banach-Steinhaus Theorem which was another famous theorem of Functional Analysis. We also discussed the famous Uniform Boundedness Principle in this chapter.

BIBLIOGRAPHY

1) B. Choudhary Sudarsan Nanda- Functional Analysis with Applications

2) Balmohan V.Limaye - Functional Analysis

3) Daniel Daners - Introduction To Functional Analysis

4) Amol Sasane - Functional Analysis and its Applications